Scattering of Water Waves in an Ocean with Uniform Porous Bed by a Surface Discontinuity Due to Inertial Surfaces in Presence of Surface Tension

Subhabrata Gangopadhyay, Uma Basu

Abstract – The phenomenon of scattering of water waves by a surface discontinuity over uniform porous sea-bottom of finite depth is investigated in the framework of linearized water wave theory. The surface discontinuity is thought of as originating due to two vast floating inertial surfaces of different materials having different densities. The inertial surfaces are considered to be subject to surface tension. The eigen function matching technique is used as the method of solution. Hydrodynamic parameters of interest such as reflection and transmission coefficients are obtained by employing residue calculus method. These coefficients are computed numerically by considering different values of the porosity parameter and for a fixed value of surface tension and the results are depicted graphically.

Keywords – eigen function matching, inertial surface, porous sea bed, reflection coefficient, residue calculus method, surface discontinuity, surface tension, transmission coefficient, water wave scattering.

1 INTRODUCTION

The problem of water wave scattering by obstacles situated at the bed in a finite depth of water has been studied in the framework of linearised theory by several scientists over a last few decades. Some of the earliest contributors in this subject were Lamb[1], Stoker[2], Kreisel[3] and Davies[4]. Basu and Mandal [5] studied the water wave scattering problem in presence of bottom undulation and surface tension in the free surface using perturbation expansion in terms of bed undulation parameter.

Another class of problem is that of water wave scattering in presence of a discontinuity at the upper surface of water. A discontinuity in the upper surface or elsewhere may occur when there is a difference of wave number of the incoming waves of certain frequency from a sudden change in the constant width of the region. Therefore, there will be two different boundary conditions on the either side of the ocean. The upper surface of the ocean may be covered by two vast sheets of floating ice plate or mat of different thickness or materials, broken ice-cover, semi-infinite floating dock etc. The presence of obstacles or materials of different densities or properties yields a change in the boundary condition due to difference in wave number. Evans and Linton [6] considered the problem of water wave scattering by a surface discontinuity in an uniform finite depth of water. They obtained the reflection and transmission coefficients by employing technique of residue calculus. Mandal and De[7] considered the problem of water wave scattering by a small undulation at the bottom in presence of upper surface discontinuity and arrived at the energy coefficients using perturbation method and Green's integral theorem.

The problem of water wave interaction over a permeable bed or porous bottom is also investigated by several researches in recent times. If the bottom is composed of some specific type of porous materials, rigid or non-rigid, the effect of porosity on the hydrodynamic coefficients is another important aspect of study. A porous sea bottom or structure may be rigid or non-rigid. The only difference is that a rigid or impermeable type porous structure or bottom does not allow the fluid to penetrate into it. The water wave interaction with the porous sea bed was studied by Chakrabarti [8], Mase and Takeba [9], Silva, Salles and Palacio [10], Jeng [11], and many others. Martha and Bora [12] applied Fourier transform to analyse scattering waves by small undulation on a porous sea-bed. The flow of fluid into the porous media or the presence of porous substances in the bottom leads to different

Subhabrata Gangopadhyay is currently an Associate Professor in the Department of Mathematics, The Bhawanipur Education Society College, 5,L.L.R.Sarani, Kolkata – 700 020, India.
E-mail: subhagan31@gmail.com
Uma Basu, is currently a Professor in the Department of Applied Mathematics, University of Calcutta, 92, A.P.C. Road, Kolkata – 700 009, India.

phenomena like energy dissipation, wave damping or decaying of wave height reaching towards the coast etc.

In the present paper, we consider the problem of scattering of an incoming wave train in presence of a discontinuity at the upper surface of the ocean. The upper surface is assumed to be covered by two semi-infinite inertial surfaces subject to surface tension. The inertial surfaces are considered to be of negligible thickness and are composed of different materials and densities. Hence, there will be a difference of wave number of the incoming wave train. If the problem is formulated mathematically, there will be two different boundary conditions on either side of the discontinuity of the upper surface of the ocean. The water is of uniform finite depth and the bed is composed of some specific kind of rigid porous material which is characterized by a known porosity parameter G'. The incoming wave train is partially reflected and partially transmitted through the ocean. The method of residue calculus of the complex variable theory (cf.[13]) is employed here to determine the reflection and transmission coefficients. Evans and Linton [6] also followed the same technique to obtain these hydrodynamic coefficients for uniform non-porous bottom of finite depth. Here, the effect of porosity on the reflection and transmission coefficients are investigated numerically and corresponding graphs are plotted against the wave number of the incident wave for different values of the porous parameter and for a fixed value of the surface tension parameter.

2 MATHEMATICAL FORMULATION

We consider the two dimensional motion in case of uniform finite depth of water. A rectangular cartesian coordinate system is chosen in which $y-$axis is taken vertically downwards in the fluid region. The discontinuity is taken at the original by assuming that the upper surface of the ocean is covered by two vast inertial surfaces of different materials and of different densities E_1, ρ and $E_2\rho$ respectively. Moreover, the inertial surfaces are subject to a fixed value of the surface tension with parameter M given by $M = T/\rho g$ where g is the acceleration due to gravity, T is the coefficient of surface tension, ρ is the density of water. The ocean bottom is composed of some specific kind of porous materials characterized by the porosity parameter G' where $G' = \frac{\alpha}{\sqrt{\nu}}$, the quantity α is dimensionless constant which depends on the structure of the porous medium and ν is the permeability of the porous medium. Let a train of surface water wave be incident from negative infinity, which is partially reflected and partially transmitted through the ocean. Assuming that the fluid flow is irrotational and the motion is simple harmonic in time t with angular frequency ω, it can be described by a velocity potential $\psi(x,y,t) = Re\{\varphi(x,y)e^{-i\omega t}\}$, where $\varphi(x,y)$ satisfies the two dimensional Laplace equation:

$$\nabla^2\varphi = 0 \text{ in the entire fluid region} \quad (1)$$

The upper surface boundary conditions taking surface tension parameter M into account, are given by:

$$K_1\varphi + \varphi_y + M\varphi_{yyy} = 0 \text{ on } y = 0, x < 0 \quad (2)$$

$$K_2\varphi + \varphi_y + M\varphi_{yyy} = 0 \text{ on } y = 0, x > 0 \quad (3)$$

This produces a discontinuity in the upper surface boundary conditions at the point (0,0) where

$K_1 = \frac{K}{1-E_1K}$, $K_2 = \frac{K}{1-E_1K}$, $E_1E_2 < \frac{g}{\omega^2}$ and $K = \frac{\omega^2}{g}$.

The edge condition is given by:

$$r^{\frac{1}{2}}\nabla\varphi = 0(1) \text{ as } r = \{x^2 + y^2\}^{\frac{1}{2}} \to 0 \quad (4)$$

The sea-bottom boundary condition is given by:

$$\varphi_y - G'\varphi = 0 \text{ on } y = h \quad (5)$$

The far field behavior of $\varphi(x,y)$ is described by:

$$\varphi(x,y) \sim \begin{Bmatrix} Te^{is_0x}\psi_0^2(y) & as\ x \to \infty \\ (e^{ik_0x} + Re^{-ik_0x})\psi_0^1(y) & as\ x \to -\infty \end{Bmatrix} \quad (6)$$

where

$\psi_0^1(y) = N_0^1\left[coshk_0(h-y) - \frac{G'}{k_0}sinhk_0(h-y)\right]$,

$\psi_0^2(y) = N_0^2\left[coshs_0(h-y) - \frac{G'}{s_0}sinhs_0(h-y)\right]$

and

$$N_0^1 = \frac{2k_0\sqrt{k_0}}{\sqrt{2k_0\left(G' - G'^2h + k_0^2h\right) - 2G'k_0cosh2k_0h + (k_0^2 + G'^2)sinh2k_0h}}$$

$$N_0^2 = \frac{2s_0\sqrt{s_0}}{\sqrt{2s_0\left(G' - G'^2h + s_0^2h\right) - 2G's_0cosh2s_0h + (s_0^2 + G'^2)sinh2s_0h}}$$

Here, $e^{ik_0x}\psi_0^1(y)$ represents the incident wave field, R and T are respectively the unknown reflection and transmission coefficients to be determined. k_0 and s_0 are the real positive roots (cf. McIver [14]) of the following two transcendental equations in terms of λ :

$$\left(\lambda + M\lambda^3 + \frac{K_1G'}{\lambda}\right)tanh\lambda h = K_1 + G' + G'M\lambda^2,$$

$$\left(\lambda + M\lambda^3 + \frac{K_2G'}{\lambda}\right)tanh\lambda h = K_2 + G' + G'M\lambda^2.$$

3 SURFACE DISCONTINUITY : ENERGY IDENTITY RELATION

As mentioned earlier, the difference in the wave number could arise due to change in the constant width of the region or sudden change in the boundary condition. The energy identity $|R|^2 + |T|^2 = 1$ is not followed in this form because of the presence of discontinuity at the upper surface boundary condition at the junction of the two inertial surfaces, i.e. at $x = 0$. However, the modified energy identity has been formulated using Green's integral theorem by Evans and Linton [6]. Here we reproduce the same energy identity in case of uniform porous bottom. Two distinct types of solutions can be considered describing waves incident from either $x \to -\infty$ or $x \to \infty$ respectively and these waves are partially reflected and partially transmitted from $x = 0$.

When the wave train is incident from negative infinity direction,

$$\varphi(x,y) \sim \begin{cases} Te^{is_0 x}\psi_0^2(y) & as\ x \to \infty \\ (e^{ik_0x} + Re^{-ik_0x})\psi_0^1(y) & as\ x \to -\infty \end{cases} \quad (7)$$

And when the wave train is incident from $x \to +\infty$

$$\chi(x,y) \sim \begin{cases} te^{is_0 x}\psi_0^2(y) & as\ x \to -\infty \\ (e^{ik_0x} + re^{-ik_0x})\psi_0^1(y) & as\ x \to \infty \end{cases} \quad (8)$$

Employing Green's integral theorem for the two functions $\chi(x,y)$ and $\varphi(x,y)$, along a contour L bounded by the lines

$y = 0, -X \le x \le X; x = \pm X, 0 \le y \le h; y = h, -X \le x \le X\ (X > 0)$

we get,

$$\oint_L (\varphi\chi_\eta - \varphi_\eta\chi)dl = 0 \quad (9)$$

where η is the outward normal to the line element dl. There is no contribution to the integral from the part $0 < y < h, x = 0$ and $y = h, -X \le x \le X\ (X > 0)$. Now using the far field conditions (7) and (8), we get,

$$\alpha|t| = |T|. \quad (10)$$

The following relations can be obtained by choosing the functions $\varphi, \bar{\varphi}$; $\bar{\chi}, \chi$ and $\varphi, \bar{\chi}$ in turn, in place of φ, χ in (9) respectively.

$$\alpha(1 - |R|^2) = |T|^2, \quad (11)$$

$$\alpha(1 - |r|^2) = |t|^2, \quad (12)$$

$$\alpha|R||r| + |T||t| = 0, \quad (13)$$

where $\alpha = \frac{k_0}{s_0}$. Now eliminating $\frac{|r|}{|t|}$, we obtain,

$$|R|^2 + \frac{1}{\alpha}|T|^2 = 1. \quad (14)$$

The above relation holds good in absence of the discontinuity at $x = 0$ in the form

$$|R|^2 + |T|^2 = 1 \quad (15)$$

which is the well known energy identity.

4 METHOD OF SOLUTION: EIGEN FUNCTION MATCHING TECHNIQUE

We consider the orghogonal depth eigen functions for the two regions ($x < 0$ and $x > 0$) respectively as:

$$\psi_n^1(y) = N_n^1\left[cosk_n(h-y) - \frac{G'}{k_n}sink_n(h-y)\right],$$

$$\psi_n^2(y) = N_n^2\left[coss_n(h-y) - \frac{G'}{s_n}sins_n(h-y)\right]$$

where

$$N_n^1 = \frac{2k_n\sqrt{k_n}}{\sqrt{2k_n\left(G' - G'^2h + k_n^2h\right) - 2G'k_n cos2k_nh + (k_n^2 + G'^2)sink_nh}}$$

$$N_n^2 = \frac{2s_n\sqrt{s_n}}{\sqrt{2s_n\left(G' - G'^2h + s_n^2h\right) - 2G's_n cos2s_nh + (s_n^2 + G'^2)sins_nh}}$$

and k_n, s_n $(n = 1,2,3 \dots)$ are given by the following two equations:

$$\left(k_n - Mk_n^3 - \frac{K_1G'}{k_n}\right)tank_nh + (K_1 + G' - G'Mk_n^2) = 0$$

$$\left(s_n - Ms_n^3 - \frac{K_2G'}{s_n}\right)tans_nh + (K_2 + G' - G'Ms_n^2) = 0.$$

The potential function $\varphi(x,y)$ can now be expanded for two different regions in terms of orthogonal depth eigen functions in the form given by

$$\varphi(x,y) = \begin{cases} \sum_{n=0}^{\infty} B_n e^{-s_nx}\psi_n^2(y) & as\ x > 0 \\ e^{ik_0x}\psi_0^1(y) + \sum_{n=0}^{\infty} A_n e^{k_nx}\psi_n^1(y) & as\ x < 0, \end{cases} \quad (16)$$

where

$$A_0 = R$$

$$B_0 = T$$

and A_n, B_n $(n = 1,2, \dots)$ are the unknown constants.

The matching conditions at $x = 0$ for $\varphi(x,y)$ and the orthogonality of the depth eigen functions produce the following two systems of linear equations:

$$\sum_{n=1}^{\infty}\frac{V_n}{s_n - k_m} = A\delta_{0m}, \quad (17)$$

$$\sum_{n=1}^{\infty}\frac{U_n}{k_n - s_m} = N_0^1\left[coshk_0(h-y) - \frac{G'}{k_0}sinhk_0(h-y)\right]\left[\frac{R_0}{ik_0 + s_m} - \frac{1}{ik_0 - s_m}\right] \quad (18)$$

where

$$V_n = B_nN_n^2\left[coss_n(h-y) - \frac{G'}{s_n}sins_n(h-y\right],$$

$$A = \frac{2ik_0}{(K_2 - K_1)N_0^1\left[coshk_0(h-y) - \frac{G'}{k_0}sinhk_0(h-y)\right]}$$

and $U_n = A_nN_n^1[cosk_n(h-y) - \frac{G'}{k_n}sink_n(h-y),$

$(m, n = 1,2 \dots)$

The unknown constants A_n, $B_n (n = 1,2, \dots)$ can be estimated numerically from the above system of linear equations (17) and (18) after truncating the infinite sum upto desired accuracy. We now proceed to determine the reflection and transmission coefficients by appropriate use of residue calculus method.

5 REFLECTION AND TRANSMISSION COEFFICIENTS:

We consider the following integral

$$J = \oint_{C_N}\frac{f(z)}{z - k_m}dz, (m = 0,1,2, \dots)$$

Here the function $f(z)$ has simple poles at $z = s_1, s_2, \dots, s_n$, simple zeros at $z = k_1, k_2, \dots, k_n$ and $f(z): 0\left(\frac{1}{\sqrt{z}}\right)$ as $z \to \infty$. Here, C_N are the sequence of circles with radius R_N which increases without bound as $N \to \infty$ avoiding the zeroes of the integral and all the poles and zeros are inside of it. Furthermore C_N must not pass through (0,0).

The Cauchy's integral formula and residue theorem gives:

$$f(k_0) = \frac{1}{2\pi i}\oint_{C_N}\frac{f(z)}{z - k_0}dz = \sum_{n=1}^{\infty}\frac{(Res(f(z))|z = s_n)}{s_n - k_0} \quad (19)$$

Assuming that $f(k_0) = -1$, we find

$$\delta_{0m} = \sum_{n=1}^{\infty}\frac{(Res(f(z))|z = s_n)}{s_n - k_0} \quad (20)$$

Now comparing (19) and (17), we obtain

$$V_n = A\,Res(f(z)|z = s_n).$$

Again, we consider the integral

$$I = \oint_{C_N} \frac{f(z)}{z + k_m} dz, (m = 0,1,2, \dots),$$

with the same property of the integrand function $f(z)$ as above. The matching conditions at $x = 0$ can be combined to give:

$$\sum_{n=1}^{\infty} \frac{V_n}{s_n + k_0} = -\frac{2ik_0 R}{(K_2 - K_1)N_0^1 \left[coshk_0(h-y) - \frac{G'}{k_0} sinhk_0(h-y)\right]} \quad (21)$$

The Cauchy's residue theorem gives for $m = 0$ and at $z = -k_0$

$$\sum_{n=1}^{\infty} \frac{V_n}{s_n + k_0} = Af(-k_0). \quad (22)$$

Comparing (21) and (22), we obtain

$$Af(-k_0) = \frac{2ik_0 R}{(K_2 - K_1)N_0^1 \left[coshk_0(h-y) - \frac{G'}{k_0} sinhk_0(h-y)\right]} \quad (23)$$

The function $f(z)$ can be taken as

$$f(z) = \frac{k_0}{z} \prod_{n=1}^{\infty} \left[\frac{\left(1 - \frac{z}{k_n}\right)\left(1 - \frac{k_0}{s_n}\right)}{\left(1 - \frac{z}{s_n}\right)\left(1 - \frac{k_0}{k_n}\right)}\right] \quad (24)$$

At $z = -k_0$, from (24), $f(z)$ gives:

$$f(-k_0) = -\prod_{n=1}^{\infty} \left[\frac{\left(1 + \frac{k_0}{k_n}\right)\left(1 - \frac{k_0}{s_n}\right)}{\left(1 + \frac{k_0}{s_n}\right)\left(1 - \frac{k_0}{k_n}\right)}\right], \quad (25)$$

and replacing A and $f(-k_0)$ in (22) we obtain:

$$R = \frac{k_0 - s_0}{k_0 + s_0} \prod_{n=1}^{\infty} \left[\frac{\left(1 + \frac{k_0}{k_n}\right)\left(1 - \frac{k_0}{s_n}\right)}{\left(1 + \frac{k_0}{s_n}\right)\left(1 - \frac{k_0}{k_n}\right)}\right].$$

Thus,

$$R = \frac{k_0 - s_0}{k_0 + s_0} e^{2i\alpha}, \quad (26)$$

where,

$$\alpha = \sum_{n=1}^{\infty} \left[tan^{-1}\left(\frac{k_0}{s_n}\right) - tan^{-1}\left(\frac{k_0}{k_n}\right)\right], (n = 1,2, \dots)$$

To obtain the transmission coefficient, we consider the following relation:

$$V_n = A\, Res(f(z)|z = s_n). \quad (27)$$

Since we have $B_0 = T$,

$V_n = B_n N_n^2 \left[coss_n(h-y) - \frac{G'}{s_n} sins_n(h-y)\right]$ and hence we obtain,

$$T = \frac{2k_0 P(s_0 + k_0)}{(K_2 - K_1)N_0^1 N_0^2 \left[coshk_0(h-y) - \frac{G'}{k_0} sinhk_0(h-y)\right]\left[coshs_0(h-y) - \frac{G'}{s_0} sinhs_0(h-y)\right]} \quad (28)$$

where,

$$P = \prod_{n=1}^{\infty} \left[\frac{\left(1 + \frac{s_0}{k_n}\right)\left(1 + \frac{s_0}{s_n}\right)}{\left(1 + \frac{k_0}{k_n}\right)\left(1 + \frac{k_0}{k_n}\right)}\right] \quad (29)$$

An alternative form of T can be obtained by using relation (14) and the expressions given by (28)-(29) as:

$$|T| = \frac{2k_0}{k_0 + s_0} \quad (30)$$

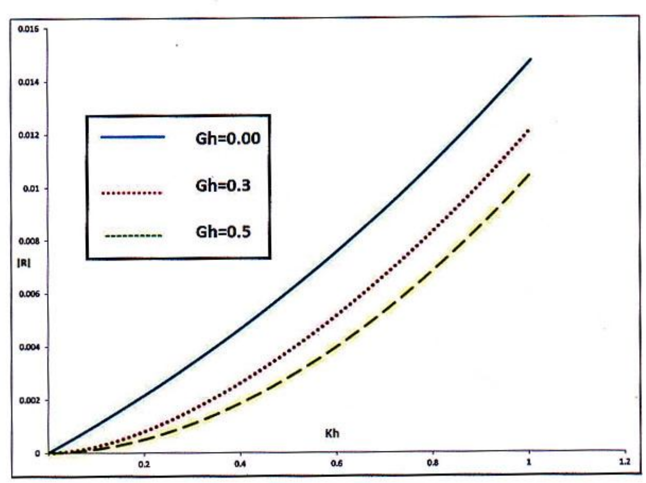
6 GRAPHS

The expressions of the reflection and transmission coefficients given by (26) and (30) are computed numerically against the wave number of the incident wave for a fixed value of surface tension. The effect of bottom porosity is investigated on the values of the reflection and transmission coefficients by considering different values of the dimensionless porosity parameter $G'h$. The absolute values of the reflection and transmission coefficients are given by:

$$R = \frac{k_0 - s_0}{k_0 + s_0}$$

$$T = \frac{2k_0}{k_0 + s_0}$$

These absolute values of the hydrodynamics coefficients are depicted in the fig.1 and fig.2 respectively for different values of the dimensionless porosity parameter $G'h = 0.00, 0.3, 0.5$ respectively. The surface tension parameter M is fixed at 0.1. In fig.1, it is seen that as the value of the dimensionless porosity parameter $G'h$ increases, the values of $|R|$ decrease rapidly. This may be attributed due to the presence of specific porous material at the bottom of the ocean which resists the wave field reflection by the discontinuity at the upper surface. The absolute values of the reflection coefficient is decreased due to the characteristic of the porous materials at the bottom. The reverse phenomena is observed in the fig.2 for the case of $|T|$. As the value of the porosity parameter $G'h$ increases, the values of $|T|$ also increase. This fact can be explained by the energy identity relation given by (14) in case of a difference in wave number. Hence the effect of porosity in the ocean bed does not violate the energy identity relation



as given by (14) in presence of upper surface discontinuity. We also note that, in absence of the upper surface discontinuity, we have $k_0 = s_0$ and in that case $R = 0, T = 1$.

ISSN 2229-5518

Fig.1: Reflection coefficient for different values of porosity parameter at a fixed M

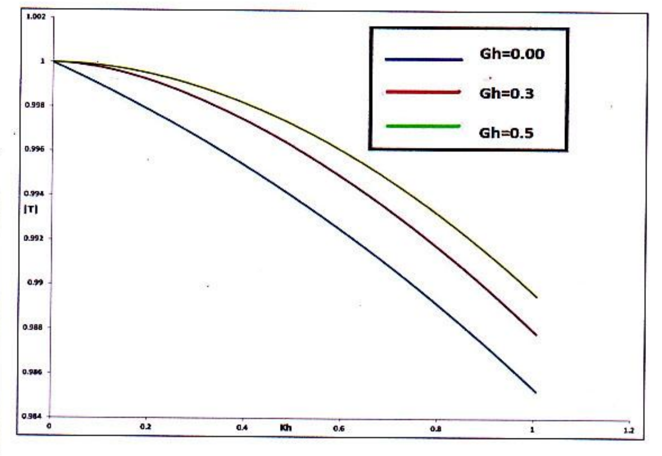


Fig.2: Transmission coefficient for different values of porosity parameter at a fixed M

7 CONCLUSION

The present study is concerned with scattering of surface wave by a discontinuity at the upper surface in a finite depth of water with porous bottom. The discontinuity arises due to presence of two types of semi-infinite inertial surfaces on either side of the origin. The inertial surfaces are also subject to surface tension. The eigen function matching technique and the residue calculus method have been made use of to determine analytical expressions of reflection and transmission coefficients. The magnitude of the hydrodynamic coefficients are plotted against wave number for different values of the porous parameter and for a fixed value of the surface tension parameter. From the analytical and numerical results, it is observed that the porous bottom has an effect on the absolute value of the hydrodynamic coefficients.

The present investigation is significant in its applications to ocean engineering, marine sciences and coastal dynamics.

REFERENCES

[1] H.Lamb, Hydrodynamics, *University Press,* 708, 1932

[2] J.J.Stoker, Water waves: The mathematical theory with applications, 1992

[3] G.Kreisel, Surface waves, *Quart. Journ. Appl. Math,* 7, 21-44, 1949

[4] A.G.Davies, The reflection of wave energy by undulation on the seabed, *Dyn. Atoms. Ocean*, 6, 207-232, 1982

[5] U.Basu, B.N.Mandal, Diffraction of water waves by a deformation of the bottom, *Indian J.Pure Appl.Math.,* 22, 781-786, 1991

[6] D.V.Evans, C.M.Linton, On step approximations for water-wave problems, *J.Fluid Mechanics.,* 278, 229-244, 1994

[7] B.N.Mandal, S.De., Surface wave propagation over small undulation at the bottom of an ocean with surface discontinuity, *Geophysical and Astrophysical Fluid Dynamics,* 103, 19-30, 2009

[8] A.Chakrabarti, A note on the porous wavemaker problem, *Acta Mechanica,* 77, 121-129, 1989

[9] H.Mase, K.Takeba, Bragg scattering of gravity waves over a porous ripple bed, *Proc. 24th ICCE Kobe (Japan),* 635-649, 1994

[10] R.Silva, P.Salles and A.Palacio, Linear wave propagating over a rapidly varying finite porous bed, *Coastal Eng.,* 44, 239-260, 2002

[11] D.S.Jeng, Wave dispersion equation in a porous sea bed, *Coastal Eng.,* 28, 1585-1599, 2001

[12] S.C.Martha, S.N.Bora, Water wave diffraction by a small deformation of the ocean bottom for oblique incidence, *Geophys. And Astro. Fluid Dyn.*, 101, 65-80, 2007

[13] J.B.Conway, Functions of one complex variable (Volume 1)

[14] P.McIver, The dispersion relation and eigen function expansion for water wave in a porous structure, *J.Eng.Math.,* 34, 319-344, 1998